

DECOMPOSITION OF JACOBIAN VARIETIES OF CURVES WITH DIHEDRAL ACTIONS VIA EQUISYMMETRIC STRATIFICATION.

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ABSTRACT. Given a compact Riemann surface X with an action of a finite group G , the group algebra $\mathbb{Q}[G]$ provides an isogenous decomposition of its Jacobian variety JX , known as the group algebra decomposition of JX . We consider the set of equisymmetric Riemann surfaces $\mathcal{M}(2n-1, D_{2n}, \theta)$ for all $n \geq 2$. We study the group algebra decomposition of the Jacobian JX of every curve $X \in \mathcal{M}(2n-1, D_{2n}, \theta)$ for all admissible actions, and we provide affine models for them. We use the topological equivalence of actions on the curves to obtain facts regarding its Jacobians. We describe some of the factors of JX as Jacobian (or Prym) varieties of intermediate coverings. Finally, we compute the dimension of the corresponding Shimura domains.

1. INTRODUCTION

Let G be a finite group acting on a given compact Riemann surface X of genus $g \geq 2$. This action induces a homomorphism $\rho : \mathbb{Q}[G] \rightarrow \text{End}_{\mathbb{Q}}(JX)$ from the rational group algebra $\mathbb{Q}[G]$ into the rational endomorphism algebra of the Jacobian variety JX of X in a natural way. The factorization of $\mathbb{Q}[G]$ into a product of simple algebras yields a decomposition of JX into abelian subvarieties [19], [16], called *the group algebra decomposition of the Jacobian variety*.

Jacobians with group action, and in particular the group algebra decomposition, have been extensively studied from different points of view in [1], [2], [7], [10], [15], [16], [17], [19], [25], [26], [28], [29], [30], [31]. Regarding dihedral groups acting on Jacobians, studies of this can be found in [6], from an algebraic point of view, and for particular actions on abelian varieties in [18].

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Since the group algebra decomposition of JX comes from algebraic data of the group G -its irreducible representations [7, 19]- this decomposition has been studied mostly, and for many years, from an algebraic point of view. However, further information about the decomposition, such as the dimensions and the polarizations of the factors, depends on the *geometry* of the action of G on X . For instance, the dimension of the factors is explicitly given in terms of the monodromy of the action [31]. Due to this dependence, it comes the general question about how the choice of the action of G on X affects the group algebra decomposition of JX . Very little is known about this question. In this work, we propose an approach through equisymmetric stratification (topological equivalence of actions).

Consider the equisymmetric stratification $\mathcal{M}_g = \cup \mathcal{M}(g, G, \theta)$ of the moduli space of Riemann surfaces of genus g defined in [5]. We use the equisymmetric stratification to obtain facts regarding Jacobian varieties of surfaces. In particular, we study the group algebra decomposition of the Jacobian variety of a curve $X \in \mathcal{M}(2n-1, D_{2n}, \theta)$, where $\mathcal{M}(2n-1, D_{2n}, \theta)$ is the equisymmetric stratum of curves of genus $2n-1$ with a D_{2n} -action and epimorphism $\theta : \Delta \rightarrow G$ determining an action σ of D_{2n} on X with signature $(0; 2, 2, 2, 2, n)$. In other words, if X, Y belong to the same stratum $\mathcal{M}(2n-1, D_{2n}, \theta)$ then the group algebra decomposition of JX and JY are the same up to a permutation of the factors. On the other hand, if X and Y belong to different strata, then the decompositions are completely different (see section 5). Besides, using the method introduced in [15] we describe the group algebra decomposition of the corresponding Jacobian varieties in terms of Jacobian (or prym) varieties of intermediate coverings. This result refines what was obtained in [6] for the action we study.

In section 2 we introduce some preliminary concepts. In section 3, we study the actions and describe the corresponding equisymmetric strata. In section 4 we find equations describing the curves in these strata as affine plane curves. In sections 5 and 6 we study the corresponding Jacobians and their loci in the moduli space of principally polarized abelian varieties. We describe the group algebra decomposition of the Jacobians, and we compute the dimension of the Shimura domain corresponding to each action.

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2. PRELIMINARIES

A compact Riemann surface corresponds to a smooth irreducible projective algebraic curve over \mathbb{C} , we say “curve” along this work referring to a compact Riemann surface. Any curve X of genus g has associated a principally polarized abelian variety $JX := H^{1,0}(X, \mathbb{C})^*/H_1(X, \mathbb{Z})$, where $H^{1,0}(X, \mathbb{C})^*$ is the dual of the complex vector space of holomorphic forms of X , and $H_1(X, \mathbb{Z})$ is the first homology group of X . This variety is called *the Jacobian variety of X* and has complex dimension g . For references about abelian varieties and Jacobians see for instance [3] and [30].

We define the group of automorphisms $\text{Aut}(X)$ of a Riemann surface X as the analytical automorphism group of X . We say that a finite group G acts on X if there is a monomorphism $\sigma : G \rightarrow \text{Aut}(X)$.

The quotient X/G (the space of orbits of the action of G on X) corresponds to a compact Riemann orbifold, we denote its genus by $\gamma = g(X/G)$. If X is uniformized by a surface Fuchsian group Γ , i.e. $X = \mathbb{H}/\Gamma$, then $X/G = \mathbb{H}/\Delta$. The canonical presentation of Δ is given by

$$(2.1) \quad \Delta = \langle \alpha_1, \beta_1, \dots, \alpha_\gamma, \beta_\gamma, x_1, \dots, x_r : x_1^{m_1} = \dots = x_r^{m_r} = \prod_{j=1}^{\gamma} [\alpha_j, \beta_j] \prod_{i=1}^r x_i = 1 \rangle.$$

Moreover, each action σ of G on X is determined by an epimorphism $\theta : \Delta \rightarrow G$ from the Fuchsian group Δ such that $\ker(\theta) = \Gamma$.

Considering the presentation of Δ as in (2.1), we define *the signature of G on X* as the vector of numbers $(\gamma; m_1, \dots, m_r)$.

If we mark every m_i with the conjugacy class of subgroups C_i of $G_i = \langle \theta(x_i) \rangle$ of G , we define *the geometric signature of G on X* as $(\gamma; [m_1, C_1], \dots, [m_r, C_r])$ [31]. The genus of X is given by the Riemann-Hurwitz formula

$$g(X) = |G|(\gamma - 1) + 1 + \frac{|G|}{2} \sum_{i=1}^r \left(1 - \frac{1}{m_i}\right).$$

A $2\gamma + r$ tuple $(a_1, \dots, a_\gamma, b_1, \dots, b_\gamma, c_1, \dots, c_r)$ of elements of G is called a *generating vector of type $(\gamma; m_1, \dots, m_r)$* if the following conditions are satisfied:

- (1) G is generated by the elements $(a_1, \dots, a_\gamma, b_1, \dots, b_\gamma, c_1, \dots, c_r)$;
- (2) $\text{order}(c_i) = m_i$; and
- (3) $\prod_{i=1}^{\gamma} [a_i, b_i] \prod_{j=1}^r c_j = 1$, where $[a_i, b_i]$ is the commutator of $a_i, b_i \in G$.

The existence of a generating vector of a given type ensures the existence of a Riemann surface with an action of the corresponding group with the given signature (Riemann's existence theorem). For a good account on Fuchsian groups and group actions see [5, 20, 21, 31, 33].

We say that two actions σ_1, σ_2 are *topologically equivalent* if there is an $\omega \in \text{Aut}(G)$ and an $h \in \text{Hom}^+(X)$ such that

$$(2.2) \quad \sigma_2(g) = h\sigma_1(\omega(g))h^{-1}$$

for all $g \in G$.

Equivalently, two epimorphisms $\theta_1, \theta_2 : \Delta \rightarrow G$ define two topologically equivalent actions σ_1, σ_2 of G on X if there exist two automorphisms, namely $\phi : \Delta \rightarrow \Delta$ and $\omega : G \rightarrow G$ such that

$$(2.3) \quad \theta_2 = \omega \circ \theta_1 \circ \phi^{-1}$$

Let \mathcal{B} be the subgroup of $\text{Aut}(\Delta)$ induced by orientation preserving homeomorphisms. In other words, σ_1, σ_2 are topologically equivalent if and only if the epimorphisms θ_1, θ_2 lie in the same $\mathcal{B} \times \text{Aut}(G)$ -class. For references see [5, 13].

Since we are interested in actions with quotient of genus 0, we consider only the elements of \mathcal{B} corresponding to compositions of automorphisms $\Phi_{i,i+1} \in \text{Aut}(\Delta)$ such that $\Phi_{i,i+1}(x_i) = x_i x_{i+1} x_i^{-1}$, $\Phi_{i,i+1}(x_{i+1}) = x_i$, $\Phi_{i,i+1}(x_j) = x_j$ for all $j \neq i, i+1$.

Let $\mathcal{M}(g, G, \theta)$ be the stratum of surfaces of genus g with automorphism group G in the conjugacy class in the mapping class group of the action determined by the epimorphism θ . Let $\overline{\mathcal{M}}(g, G, \theta)$ be the subset of surfaces having an automorphism group containing the automorphism group G in the conjugacy class determined by θ in the mapping class group $\text{Mod}(\Gamma)$.

It is known that if the signature of the action is $(\gamma; m_1, \dots, m_r)$, then the Teichmüller dimension is $3(\gamma - 1) + r$. For references about Teichmüller and moduli spaces see [5, 13, 21, 23].

Let G be a finite group. If $\psi : G \rightarrow \text{GL}(V)$ is an irreducible representation of G over \mathbb{C} afforded by the complex vector space V , we denote by F its field of definition and by K the field obtained by extending \mathbb{Q} by the values of its character χ_V ; then $K \subseteq F$ and $m_V = [F : K]$ is the Schur index of V .

If H is a subgroup of G , $\text{Ind}_H^G 1$ denotes the representation of G induced by the trivial representation of H . Besides $\langle U, V \rangle$ denotes the usual inner product of the characters. By Frobenius Reciprocity $\langle \text{Ind}_H^G 1, V \rangle = \dim_{\mathbb{C}} \text{Fix}_H V$, where $\text{Fix}_H V$ is the subspace of V fixed

under H . We use indistinctly V or χ_ψ to denote the representation or the vector space, affording it, when the context is clear.

A good account about representation theory of finite groups may be found in [9, 14, 32].

Given a compact Riemann surface X with an action of a group G we consider the induced homomorphism $\rho : \mathbb{Q}[G] \rightarrow \text{End}_{\mathbb{Q}}(JX)$.

$\mathbb{Q}[G]$ decomposes into a product $Q_0 \times \cdots \times Q_t$ of simple \mathbb{Q} -algebras, the simple algebras Q_i are in bijective correspondence with the rational irreducible representations of G . That is, for any rational irreducible representation \mathcal{W}_i of G there is a uniquely determined central idempotent e_i generating Q_i . Moreover, the decomposition of every $Q_i = L_1 \times \cdots \times L_{n_i}$ into a product of minimal left ideals (all isomorphic) gives a further decomposition of the Jacobian. There are idempotents $f_{i1}, \dots, f_{in_i} \in Q_i$ such that $e_i = f_{i1} + \cdots + f_{in_i}$ where $n_i = \dim V_i / m_{V_i}$, with V_i a \mathbb{C} -irreducible representation associated to \mathcal{W}_i . The factor $B_i^{n_i}$ is defined as the image of $m\rho(e_i) \in \text{End}(JX)$, and it is shown that it does not depend on m (up to isogeny). In the same way the factor B_i corresponds to the image of any of the f_{in_i} . See [3, 19] for details regarding these decompositions.

It is known that the factor associated to the trivial representation of G , i.e. the image of $e_0 \in Q_0$, is isogenous to $J_G = J(X/G)$. The addition map is an isogeny [3, 7, 19].

$$(2.4) \quad \nu : J_G \times B_1^{n_1} \times \cdots \times B_t^{n_t} \rightarrow JX.$$

This is called *the group algebra decomposition of JX* . We denote the isogeny ν by $JX \sim J_G \times B_1^{n_1} \times \cdots \times B_t^{n_t}$.

2.1. Regarding the factors. Consider the decomposition of JX given in (2.4) and $H \leq G$, then the corresponding group algebra decomposition of the Jacobian variety $J_H = J(X/H)$ of the intermediate quotient X/H is given as follows

$$(2.5) \quad J_H \sim J_G \times B_1^{\frac{\dim \text{Fix}_H V_1}{m_1}} \times \cdots \times B_t^{\frac{\dim \text{Fix}_H V_t}{m_t}}$$

As in [7], for every $H \leq G$ define $p_H = \frac{1}{|H|} \sum_{h \in H} h$. It is a central idempotent in $\mathbb{Q}[H]$ corresponding to the trivial representation of H . Moreover, $\text{Im}(p_H) = \pi_H^*(J_H)$, where $\pi_H^*(J_H)$ is the pullback of J_H by $\pi_H : X \rightarrow X/H$. Finally define $f_H^i = p_H e_i$, which is an idempotent in $\mathbb{Q}[G]e_i$, if $\dim \text{Fix}_H V_i \neq 0$ then

$$(2.6) \quad \text{Im}(f_H^i) = B_i^{\frac{\dim \text{Fix}_H V_i}{m_i}}.$$

The induced action of G on JX provides geometrical information about the components of the group algebra decomposition of JX [7].

In fact, the dimension of the subvarieties in the decomposition (2.4) are obtained using the generating vector of the action [31, Theorem 5.2].

Theorem 2.1 (Dimension of the B_j 's). *Let G be a finite group acting on a compact Riemann surface X with geometric signature given by $(\gamma; [m_1, C_1], \dots, [m_r, C_r])$. Then the dimension of any subvariety B_i associated to a non trivial rational irreducible representation \mathcal{W}_i in (2.4), is given by*

$$\dim B_i = k_i \left(\dim V_i(\gamma - 1) + \frac{1}{2} \sum_{k=1}^r (\dim V_i - \dim \text{Fix}_{G_k} V_i) \right),$$

where G_k is a representative of the conjugacy class C_k , $\dim V_i$ is the dimension of a complex irreducible representation V_i associated to \mathcal{W}_i , $K_i = K_{V_i}$ and $k_i = m_i[K_i : \mathbb{Q}]$.

To identify the factors as Jacobians of intermediate coverings we use the following result [15, Lemma 1].

Lemma 2.2. *Let X be a Riemann surface with an action of a finite group G such that the genus of X/G is equal to zero. Consider ν the group algebra decomposition of JX as in 2.4.*

- (i) *If $H \leq G$ is such that $\dim_{\mathbb{C}} \text{Fix}_H V_i = m_i$, where m_i is the Schur index of the representation V_i , then we have that $\text{Im}(f_H^i) = B_i$. In addition,*
- (ii) *if $\dim_{\mathbb{C}} \text{Fix}_H V_l = 0$ for all $l, l \neq i$, such that $\dim_{\mathbb{C}} B_l \neq 0$ in the isotypical decomposition of JX in Equation (2.4), then*

$$J_H \sim \text{Im}(p_H) = B_i.$$

2.2. Rational irreducible representations of the dihedral group.

Let $D_{2n} = \langle a, s : a^{2n} = s^2 = (as)^2 = 1 \rangle$ be the dihedral group of order $4n$. It is known (see for instance [6, 14]) that D_{2n} has four complex irreducible representations of degree one denoted by $\chi_0, \chi_1, \chi_2, \chi_3$, and $n - 1$ of degree two ψ_j for $1 \leq j \leq n - 1$ given by

$$\psi_j(a) = \begin{pmatrix} \omega^j & 0 \\ 0 & \omega^{-j} \end{pmatrix} \text{ and } \psi_j(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where $\omega = \omega_{2n} = \exp \frac{\pi i}{n}$. The character table is in Table 1

Conjugate classes in D_{2n} :	1	a^n	$a^r (1 \leq r \leq n - 1)$	s	as
χ_0	1	1	1	1	1
χ_1	1	1	1	-1	-1
χ_2	1	$(-1)^n$	$(-1)^r$	1	-1
χ_3	1	$(-1)^n$	$(-1)^r$	-1	1
χ_{ψ_j}	2	$2(-1)^j$	$\omega^{jr} + \omega^{-jr}$	0	0

TABLE 1. Character Table of D_{2n} .

We are interested in the rational irreducible representations of D_{2n} .
Let

$$\Omega(2n) = \{d : d \text{ is a divisor of } 2n \text{ and } d < n\}.$$

For each d in $\Omega(2n)$, consider ψ_d as above. Its field of definition coincides with the field generated by its characters. It means that the Schur index $m_d = 1$ and

$$K_{\psi_d} = \mathbb{Q}(\omega_{2n}^d + \omega_{2n}^{-d}) = \mathbb{Q}(\omega_{\frac{2n}{d}} + \omega_{\frac{2n}{d}}^{-1}).$$

Furthermore $[K_{\psi_d} : \mathbb{Q}] = \frac{\varphi(\frac{2n}{d})}{2}$, where φ denotes the Euler function. Hence, the rational irreducible representation W_{ψ_d} of D_{2n} satisfies

$$K_{\psi_d} \otimes_{\mathbb{Q}} W_{\psi_d} = \bigoplus_{\sigma \in \text{Gal}(K_{\psi_d}/\mathbb{Q})} \psi_d^{\sigma}$$

and it is of degree $\deg(W_{\psi_d}) = \dim \psi_d [K_{\psi_d} : \mathbb{Q}] = 2 \frac{\varphi(\frac{2n}{d})}{2} = \varphi(\frac{2n}{d})$.

Remark 2.3. Observing Table 1 of the complex characters of D_{2n} , we notice that the characters χ_2 and χ_3 have the same values for each conjugation class except for the classes $\mathcal{C}_1 = \{\langle a^{2l+1}s \rangle : 0 \leq l \leq n-1\}$ and $\mathcal{C}_2 = \{\langle a^{2l}s \rangle : 0 \leq l \leq n-1\}$ represented by the elements as and s respectively. Hence, in particular we have $\chi_2(s) = \chi_3(as)$. In fact, we have the following results

- **Result 1.** $\chi_2 = \chi_3 \circ \omega$ where $\omega \in \text{Out}(D_{2n}) \subset \text{Aut}(D_{2n})$ is the outer automorphism given by $\omega(a) = a, \omega(s) = as$.
- **Result 2.** Let $\mathcal{C}_1, \mathcal{C}_2$ be the conjugation classes defined above, and ω the automorphism in Result 1, then $\text{Fix}_H \chi_2 = \text{Fix}_{\omega(H)} \chi_3$ for all $H \in \mathcal{C}_2$, where $\text{Fix}_H \chi$ is the fixed space defined in Section 2. In particular, $\text{Fix}_{\langle s \rangle} \chi_2 = \text{Fix}_{\langle \omega(s) \rangle} \chi_3$.

3. ON ACTIONS OF D_{2n} WITH SIGNATURE $(0; 2, 2, 2, 2, n)$.

We recall for this section that two actions σ_1, σ_2 of G on X are topologically equivalent if they satisfy equation (2.2) i.e. if the epimorphisms $\theta_1, \theta_2 : \Delta \rightarrow G$, lie in the same $\mathcal{B} \times \text{Aut}(G)$ -class as explained in the Section 2.

In this section, we classify the different generating vectors of type $(0; 2, 2, 2, 2, n)$, i.e. we classify all the 5-tuples (c_1, \dots, c_5) of elements of D_{2n} such that D_{2n} is generated by c_1, \dots, c_5 with $\text{order}(c_i) = 2, 1 \leq i \leq 4$, $\text{order}(c_5) = n$, and $\prod_{i=1}^5 c_i = 1$. The results depend on the parity of n , thus we treat both cases separately.

3.1. n an odd number.

Proposition 3.1. *Let $n \geq 3$ be an odd number. Then, every generating vector of D_{2n} of type $(0; 2, 2, 2, 2, n)$ belongs to one of the following classes with representatives*

- I. (a^n, a^n, s, a^2s, a^2) ,
- II. (s, s, as, a^3s, a^2) .

Proof. Consider an action σ of $D_{2n} = \langle a, s | a^{2n} = s^2 = (as)^2 = 1 \rangle$ with monodromy $\theta : \Delta \rightarrow D_{2n}$, $s(\Delta) = (0; 2, 2, 2, 2, n)$, with generating vector $(\theta(x_1), \theta(x_2), \theta(x_3), \theta(x_4), \theta(x_5))$. First of all we can assume, up to an automorphism of D_{2n} , that $\theta(x_5) = a^2$. Now as θ is an epimorphism the condition $\theta(x_1x_2x_3x_4x_5) = 1$ obliges to one of the following cases:

- I. Two of x_1, x_2, x_3, x_4 are mapped to a^n and the other two to $a^{t_1}s$ and $a^{t_2}s$ such that $t_1 - t_2 \equiv 2 \pmod{(2n)}$, or
- II. All x_1, x_2, x_3, x_4 are mapped to $a^{t_i}s$, $1 \leq i \leq 4$, such that $t_1 - t_2 + t_3 - t_4 \equiv 2 \pmod{(2n)}$.

Observe that in case I the parity of t_1 and t_2 coincides. Applying a suitable composition of automorphisms of Δ we obtain that a generating vector of an action σ in this case is in principle of the following types

- i) $(a^n, a^n, a^{t_1}s, a^{t_2}s, a^2)$, with t_1 and t_2 even,
- ii) $(a^n, a^n, a^{t_1}s, a^{t_2}s, a^2)$, with t_1 and t_2 odd,

For instance $\Phi_{2,3} \in \mathcal{B}$ conjugates the generating vector (a^n, s, a^n, a^2s, a^2) with (a^n, a^n, s, a^2s, a^2) , since $\Phi_{2,3}(x_2) = x_2x_3x_2^{-1}$, $\Phi_{2,3}(x_3) = x_2$ and $\Phi_{2,3}(x_i) = x_i$ for $i \neq 2, 3$.

Finally applying an inner automorphism of D_{2n} one gets that any generating vector of type i) is equivalent to (a^n, a^n, s, a^2s, a^2) and any generating vector of type ii) is equivalent to $(a^n, a^n, as, a^3s, a^2)$. Using an outer automorphism of D_{2n} , such as $s \mapsto as$, one gets that any generating vector in this case is equivalent to (a^n, a^n, s, a^2s, a^2) .

In case II at least one of t_i must be odd and one even, and the parity of $t_1 + t_3$ equals the one of $t_2 + t_4$, these facts force the exponents t_i to be two odd and two even. This is,

$(a^{t_1}s, a^{t_2}s, a^{t_3}s, a^{t_4}s, a^2)$, with two t_i 's even numbers, and two t_i 's odd.

We proceed in an analogous way as in case I. For instance to prove that (s, s, as, a^3s, a^2) is equivalent to (as, as, s, a^2s, a^2) , hence they correspond to the same action (up to topological equivalence), we use the element $\Phi_{2,3}^{-1}\Phi_{1,2}^{-1}\Phi_{3,4}\Phi_{2,3}^{-1} \in \mathcal{B}$. The element $\Phi_{2,3}^{-1}\Phi_{1,2}^{-1}\Phi_{3,4}\Phi_{1,2}\Phi_{2,3}^{-1} \in \mathcal{B}$

conjugates $(as, as, a^2s, a^4s, a^2)$ to (as, as, s, a^2s, a^2) , and the element $\Phi_{2,3}\Phi_{1,2}^{-1}\Phi_{3,4}^{-1}\Phi_{2,3}^{-1} \in \mathcal{B}(s, s, as, a^3s, a^2)$ to $(as, as, a^2s, a^4s, a^2)$.

Finally, we obtain that any generating vector in this case is equivalent to (s, s, as, a^3s, a^2) .

To prove that σ_1 is not equivalent to σ_2 , it is enough to note that $\langle a^n \rangle$ is a characteristic subgroup of D_{2n} . □

Definition 3.2. Let n be an odd number. Let σ be an action of D_{2n} on a curve with signature $(0; 2, 2, 2, 2, n)$. We say that σ is an action of

- (i) Type 1, we write σ_1 , if its generating vector is equivalent to (a^n, a^n, s, a^2s, a^2) .
- (ii) Type 2, we write σ_2 , if its generating vector is equivalent to (s, s, as, a^3s, a^2) .

A direct consequence from Proposition 3.1 is that there are two non equivalent topological actions in correspondence with the two classes of generating vectors. Thus, there are two conjugacy classes of D_{2n} in the mapping class group $\text{Mod}(\Gamma_{2n-1})$ for n odd.

We summarize all the results in this subsection in Theorem 3.3.

Theorem 3.3. *The family of compact Riemann surfaces with action of D_{2n} of type $(0; 2, 2, 2, 2, n)$ has two equisymmetric strata each one of dimension 2: $\mathcal{M}(2n-1, D_{2n}, \theta_1)$ and $\mathcal{M}(2n-1, D_{2n}, \theta_2)$, where θ_1, θ_2 are the epimorphisms corresponding to the actions σ_1, σ_2 in Definition 3.2.*

3.2. n an even number.

Proposition 3.4. *Let $n \geq 2$ be an even number. Then, every generating vector of D_{2n} of type $(0; 2, 2, 2, 2, n)$ belongs to the class represented by (s, s, as, a^3s, a^2) . Hence there is just one action σ up to topological equivalence.*

Proof. We proceed as in Proposition 3.1, except that here we obtain just one equivalence class of generating vectors; the vector $(a^n, a^n, as, a^3s, a^2)$ does not generate D_{2n} for even n since the center of D_{2n} is cyclic of order 2 and the center of $D_n \times \mathbb{Z}_2$ is isomorphic to the Klein group. The last statement follows from the fact that topologically non equivalent actions are in correspondence with orbits of generating vectors. □

We summarize the results in this subsection in the following theorem.

Theorem 3.5. *Let n be an even number. Then the family of compact Riemann surfaces with action of D_{2n} of type $(0; 2, 2, 2, 2, n)$ has one equisymmetric stratum of dimension 2: $\mathcal{M}(2n-1, D_{2n}, \theta)$ where θ is the epimorphism corresponding to the action σ in Proposition 3.4.*

4. EQUATION FOR THE CURVES.

We want to find equations for the curves X with the action(s) of D_{2n} with signature $(0; 2, 2, 2, 2, n)$. From the theory developed before (Section 3), we know that the actions correspond to two strata in the case n odd with generating vectors (a^n, a^n, s, a^2s, a^2) and (s, s, as, a^3s, a^2) , for the actions labeled σ_1 and σ_2 respectively (see Definition 3.2), and one stratum for n even with generating vector (s, s, as, a^3s, a^2) (see Proposition 3.4). We recall that each action determines a 2-dimensional equisymmetric stratum, hence our planar affine models will depend on two complex parameters.

Theorem 4.1. *Let G be the dihedral group $D_{2n} = \langle a^{2n} = s^2 = (as)^2 = 1 \rangle$ acting on genus $g = 2n - 1$ with signature $(0; 2, 2, 2, 2, n)$. If n is odd, denote by $\mathcal{M}(g, G, \theta_1)$ and $\mathcal{M}(g, G, \theta_2)$ the two 2-dimensional equisymmetric strata corresponding to the actions σ_1 and σ_2 . If n is even, denote by $\mathcal{M}(g, G, \theta_2)$ the 2-dimensional equisymmetric stratum corresponding to the (unique) action σ_2 . Then*

- (1) *The Riemann surfaces in $\mathcal{M}(g, G, \theta_1)$ are hyperelliptic curves, and an affine plane model representing them is given by*

$$\mathcal{H}_{\lambda, \mu} : y^2 = (x^n - \lambda^n)(x^n - \frac{1}{\lambda^n})(x^n - \mu^n)(x^n - \frac{1}{\mu^n}),$$

with $\lambda, \mu \in \mathbb{C} \setminus \{0\}$ and $\lambda^{2n}, \mu^{2n} \neq 1$.

- (2) *The Riemann surfaces in $\mathcal{M}(g, G, \theta_2)$ are elliptic n -gonal curves. An affine plane model representing them is given by*

$$\mathcal{E}_{a,b} : x^{2n} + y^{2n} + ax^ny^n + bx^n + by^n + 1 = 0,$$

with $a, b \in \mathbb{C} \setminus \{0\}$.

Proof. Let X be a Riemann surface in the strata $\mathcal{M}(g, G, \theta_1)$, and consider the central subgroup $H = \langle a^n \rangle \leq D_{2n}$ of order 2. Since n is odd in this case, we have $G = D_{2n} = D_n \times H$. A generating vector for the actions in this strata is (a^n, a^n, s, a^2s, a^2) , then we have an epimorphism $\theta_1 : \Delta \rightarrow G$ defining the action σ_1 . Let $\Lambda = \theta_1^{-1}(H)$. Since H is a normal subgroup of G , hence $N = \Delta/\Lambda \cong D_{2n}/H \cong D_n$. Following [20, 33], we compute the ramification of the covering $X \rightarrow X/H$. This is, for every generating element x_i in the canonical presentation (2.1) of Δ we determine the order n_i of the class $\overline{x_i}$ in N , thus in the quotient it contributes with $2n/n_i$ points with branching number m_i/n_i , where m_i is the order of x_i in Δ . For the action σ_1 we have

- (1) $\text{order}(\overline{x_1}) = \text{order}(\overline{x_2}) = \text{order}(\overline{a^n}) = 1$ in N . Therefore each one contributes with $2n$ points of branching number 2.
- (2) $\text{order}(\overline{x_3}) = \text{order}(\overline{s}) = \text{order}(\overline{a^2s}) = \text{order}(\overline{x_4}) = 2$ in N . Therefore each one contributes with n regular points.
- (3) $\text{order}(\overline{x_5}) = \text{order}(\overline{a^2}) = n$ in N . Therefore it contributes with two regular points.

Summarizing, the covering $f : X \rightarrow Y = X/H$ has $4n$ branch points. By Riemann-Hurwitz we obtain that the quotient surface $Y = X/H$ is of genus 0, hence X is hyperelliptic and the group $G/H \cong D_n$ acts on the branch points having two orbits of $2n$ points each.

We apply the results in [35] or [27], take one representative λ and μ for each orbit and set $w = e^{2\pi i/n}$. Then the full orbit of each point is $\{\lambda, w\lambda, w^2\lambda, \dots, w^{n-1}\lambda, w\lambda^{-1}, w^2\lambda^{-1}, \dots, w^{n-1}\lambda^{-1}\}$ and analogously $\{\mu, w\mu, w^2\mu, \dots, w^{n-1}\mu, w\mu^{-1}, w^2\mu^{-1}, \dots, w^{n-1}\mu^{-1}\}$.

Therefore one equation for X is

$$y^2 = \prod_{k=0}^{n-1} (x - w^k \lambda) \prod_{k=0}^{n-1} (x - \frac{w^k}{\lambda}) \prod_{k=0}^{n-1} (x - w^k \mu) \prod_{k=0}^{n-1} (x - \frac{w^k}{\mu}),$$

the result follows from this.

For surfaces given by $\theta_2 : \Delta \rightarrow G$, i.e. with action σ_2 and generating vector (s, s, as, a^3s, a^2) , consider the subgroups $H = \langle a^2 \rangle$ and $\Lambda = \theta_2^{-1}(H)$ of index 4 in G respectively Δ . As before, let $N = \Delta/\Lambda \cong D_{2n}/H$. Again by [20, 33] we have

- (1) $\text{order}(\overline{x_1}) = \text{order}(\overline{x_2}) = \text{order}(\overline{s}) = 2$ in N . Therefore each one contributes with regular points.
- (2) $\text{order}(\overline{x_3}) = \text{order}(\overline{as}) = \text{order}(\overline{a^3s}) = \text{order}(\overline{x_4}) = 2$ in N . Therefore each one contributes with two regular points.
- (3) $\text{order}(\overline{x_5}) = \text{order}(\overline{a^2}) = 1$ in N . Therefore it contributes with 4 conic points with branch number p .

Using Riemann-Hurwitz to compute the genus γ of the quotient $Y = X/H$ we get $2n - 1 = n(\gamma - 1) + 1 + \frac{1}{2}(4(n - 1))$, then $\gamma = 1$, hence X is a so called elliptic n -gonal curve. An affine plane model for this kind of curves is

$$x^{2n} + y^{2n} + ax^n y^n + bx^n + by^n + 1 = 0,$$

with $a, b \in \mathbb{C} \setminus \{0\}$.

□

Remark 4.2. (1) Notice that for $n = 2$ we only have the stratum determined by σ_2 , and the corresponding Riemann surfaces are

of genus 3. This locus corresponds to the one described in the second row of [22, Table 2].

- (2) Notice that $\mathcal{E}_{0,0} \in \overline{\mathcal{M}}(g, G, \theta_2)$ has a larger automorphism group. In fact $\mathcal{E}_{0,0}$ is the Fermat curve $F_{2n} : x^{2n} + y^{2n} + 1 = 0$, which has $\text{Aut}(F_{2n}) = (\mathbb{Z}_{2n} \times \mathbb{Z}_{2n}) \rtimes S_3$.

5. GROUP ALGEBRA DECOMPOSITION OF JACOBIANS OF CURVES WITH D_{2n} -ACTION AND SIGNATURE $(0; 2, 2, 2, 2, n)$

Sections 3 and 4 were devoted to study curves with our actions of D_{2p} , and their strata in the moduli space of Riemann surfaces. In the following sections, 5 and 6, we develop results concerning the corresponding Jacobian varieties and their Shimura domains in the Siegel upper half space. In particular, in this section we study the group algebra decomposition of the Jacobians associated to these actions.

5.1. n an odd number. Let us consider as before $D_{2n} = \langle a, s : a^{2n} = s^2 = (as)^2 = 1 \rangle$ acting on a curve X of genus $g = 2n - 1$ with signature $(0; 2, 2, 2, 2, n)$. From Section 2, we know that D_{2n} has four complex irreducible representations of degree one and $n - 1$ of degree two. All of them with Schur index equal to 1.

To make further calculations easier, we define

$$(5.1) \quad \Omega(2n) = \Omega_{\text{odd}} \cup \Omega_{\text{even}},$$

with $\Omega_{\text{odd}} = \{d \in \Omega(2n) : d \text{ is an odd number}\}$, $\Omega_{\text{even}} = \Omega(2n) \setminus \Omega_{\text{odd}}$.

The rational irreducible representations of D_{2n} are $\chi_0, \chi_1, \chi_2, \chi_3$ of degree one, and

$$\mathcal{W}_d := \mathcal{W}_{\psi_d} = \bigoplus_{\sigma \in \text{Gal}(K_{\psi_d}/\mathbb{Q})} \psi_d^\sigma,$$

for all $d \in \Omega(2n)$.

Then the group algebra decomposition of JX is given by

$$JX \sim E_0 \times E_1 \times E_2 \times E_3 \times \prod_{d \in \Omega_{\text{odd}}} B_d^2 \times \prod_{d \in \Omega_{\text{even}}} B_d^2,$$

where E_i is a subvariety of JX associated to χ_i , and $B_0 = J(X/D_{2n})$ has dimension 0 in our case.

To compute the dimensions of these subvarieties we use Theorem 2.1; in this case we get

$$(5.2) \quad \dim E_i = -\dim \chi_i + \frac{1}{2} \sum_{k=1}^5 (\dim \chi_i - \dim \text{Fix}_{G_k} \chi_i),$$

$$\dim B_d = \frac{1}{2}\varphi\left(\frac{2n}{d}\right)\left(-\dim \psi_d + \frac{1}{2}\sum_{k=1}^5(\dim \psi_d - \dim \text{Fix}_{G_k}\psi_d)\right)$$

where $0 \leq i \leq 3$ and $d \in \Omega(2n)$.

Therefore we need the dimensions of $\text{Fix}_{G_k}\chi_i$ and $\text{Fix}_{G_k}V_d$, which are obtained in Proposition 5.1.

Proposition 5.1. *The dimension of the spaces $\text{Fix}_H\chi_i$ and $\text{Fix}_{G_k}V_d$ for every H non-trivial cyclic subgroup of G is given in Table 2.*

Subgroups, $r \in \mathbb{N}$	χ_0	χ_1	χ_2	χ_3	$\psi_{\{d:d \in \Omega_{\text{even}}\}}$	$\psi_{\{d:d \in \Omega_{\text{odd}}\}}$
$\langle a^n \rangle$	1	1	0	0	2	0
$\langle a^{2r} \rangle$	1	1	1	1	0	0
$\langle a^{2r+1} \rangle$	1	1	0	0	0	0
$\langle s \rangle$	1	0	1	0	1	1
$\langle a^{2r+1}s \rangle$	1	0	0	1	1	1
$\langle a^{2r}s \rangle$	1	0	1	0	1	1

TABLE 2. Dimension of fixed spaces

Proof. We want to calculate the dimension of $\text{Fix}_H V$ for V a complex irreducible representation associated to χ_i and \mathcal{W}_j , for all i and all j , and all cyclic subgroup H of G . Observe that for the representations of degree 1; χ_0, χ_1, χ_2 and χ_3 , we only need to check in the character table written before if it is equal to 1 or -1 . Then, if $H = \langle h \rangle$

$$\dim \text{Fix}_H \chi_i = \begin{cases} 1 & \text{if } \chi_i(h) = 1 \\ 0 & \text{if } \chi_i(h) = -1 \end{cases}$$

for all $0 \leq i \leq 3$. Hence, we obtain the first fourth columns of Table 2.

Now, for the representations of degree 2, we have

$$\psi_d(a^n) = \begin{pmatrix} \omega^{nj} & 0 \\ 0 & \omega^{-nd} \end{pmatrix} = \begin{pmatrix} (-1)^d & 0 \\ 0 & (-1)^d \end{pmatrix},$$

Then when d is odd, the matrix $\psi_d(a^n) = -I$ and the unique eigenvalue is $\lambda = -1$, then $\dim \text{Fix}_{\langle a^n \rangle} \psi_d = 0$. Now, on the other hand when d is even, the matrix $\psi_d(a^n) = I$ and the unique eigenvalue is $\lambda = 1$, then $\dim \text{Fix}_{\langle a^n \rangle} \psi_d = 2$.

Now for computing the dimension of the fixed spaces by $\langle a^r \rangle$, with $r \neq 0, n$, consider

$$\psi_d(a^r) = \begin{pmatrix} \omega^{rd} & 0 \\ 0 & \omega^{-rd} \end{pmatrix},$$

for all d . Then in any case the eigenvalues are $\lambda = \omega^{rd}, \omega^{-rd}$, then $\dim \text{Fix}_{\langle a^r \rangle} \psi_d = 0$.

To compute the corresponding dimension for $\langle s \rangle$ consider

$$\psi_d(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then the eigenvalues are $\lambda = 1, -1$, hence $\dim \text{Fix}_{\langle s \rangle} \psi_d = 1$ for all d . Using the same methods for $\langle a^{2r}s \rangle$ and $\langle a^{2r+1}s \rangle$, we complete the table. \square

To know whether a Jacobian variety has elliptic factors has been deeply studied [10, 16, 25, 26]. We prove in the following theorem that the Jacobian of a curve with action of D_{2n} and signature $(0; 2, 2, 2, 2, n)$ has always *at least* one elliptic factor in its group algebra decomposition.

Theorem 5.2. *If X is a Riemann surface with action of the group D_{2n} and signature $(0; 2, 2, 2, 2, n)$ as in Theorem 3.3, then JX has always (at least) one elliptic factor in its group algebra decomposition. In fact, the decompositions are*

- (i) for the action σ_1 , $JX \sim E_2 \times \prod_{\{d \in \Omega_{\text{odd}}\}} B_d^2$
- (ii) for σ_2 , $JX \sim E_1 \times \prod_{\{d \in \Omega_{\text{odd}}\}} B_d^2 \times \prod_{\{d \in \Omega_{\text{even}}\}} B_d^2$

where the subindex show the rational representation acting on the factor and E_1, E_2 are elliptic curves.

Moreover, the dimensions of the factors are given in general by

Actions	E_1	E_2	$B_{\{d \in \Omega_{\text{even}}\}}$	$B_{\{d \in \Omega_{\text{odd}}\}}$
σ_1	0	1	0	$\varphi(\frac{2n}{d})$
σ_2	1	0	$\frac{1}{2}\varphi(\frac{2n}{d})$	$\frac{1}{2}\varphi(\frac{2n}{d})$

TABLE 3. DIMENSIONS

Proof. We obtain Table 3 as direct combination of Equations (5.2) and Proposition 5.1. \square

Remark 5.3. Notice that if we consider the generating vectors (a^n, a^n, s, a^2s, a^2) and $(a^n, a^n, as, a^3s, a^2)$ corresponding to the same action σ_1 on X (see Definition 3.2), we obtain *essentially* the same decomposition of JX . The difference is the rational representation acting on the elliptic factor at the group algebra decomposition of JX . This is explained in Remark 2.3.

On the other side, when we consider topologically non-equivalent actions, σ_1 and σ_2 , we get decompositions of JX with completely different behaviour: the factors have different geometry; dimension, structure and so on.

Using Lemma 2.2 introduced in [15], we have a criteria to isolate some products of the factors in the decomposition to write them as

Jacobians (or Pryms) of intermediate coverings. We recall here the results and notation introduced in subsection 2.1.

Theorem 5.4. *Let JX be as in Theorem 5.2. Then JX decomposes as*

- (i) for the action σ_1 , $JX \sim J(X/\langle a^2 \rangle) \times J(X/\langle as \rangle)^2$,
- (ii) for σ_2 , $JX \sim J(X/\langle a^2 \rangle) \times J(X/\langle a^n, s \rangle)^2 \times P(X_{\langle s \rangle}/X_{\langle a^n, s \rangle})^2$.

Proof. Table 2 in Proposition 5.1 contains the dimension of the fixed space of cyclic for all complex representations. We give the dimension for other subgroups in Table 4.

Subgroups	χ_0	χ_1	χ_2	χ_3	$\psi_{\{d \in \Omega_{\text{even}}\}}$	$\psi_{\{d \in \Omega_{\text{odd}}\}}$
$\langle a^n, s \rangle$	1	0	0	0	1	0
$\langle a^2, s \rangle$	1	0	1	0	0	0
$\langle a^2, as \rangle$	1	0	0	1	0	0

TABLE 4. Dimension of fixed spaces.

By Theorem 5.2 we know that

- (i) σ_1 ; $JX \sim E_2 \times \prod_{\{d \in \Omega_{\text{odd}}\}} B_d^2$
- (ii) σ_2 ; $JX \sim E_1 \times \prod_{\{d \in \Omega_{\text{even}}\}} B_d^2 \times \prod_{\{d \in \Omega_{\text{odd}}\}} B_d^2$

By Lemma 2.2, to write E_2 as a Jacobian of X/H with $H \leq G$, we notice that in the table above joined with table in Proposition 5.1 that $\dim \text{Fix}_H \chi_2 = 1$ and $\dim \text{Fix}_H \psi_{d \in \Omega_{\text{odd}}} = 0$ when H is a subgroup of G in the class of subgroups represented by $\langle a^2 \rangle, \langle a^2, s \rangle$.

To write E_1 as a Jacobian of X/H with $H \leq G$, we notice that $\dim \text{Fix}_H \chi_1 = 1$ and $\dim \text{Fix}_H \psi_d = 0$, for $d \in \Omega(2n)$ when H is a subgroup of G in the class of subgroups represented by $\langle a^2 \rangle, \langle a \rangle$.

To write the other piece in the decomposition corresponding to σ_1 , $\prod_{\{d \in \Omega_{\text{odd}}\}} B_d^2$, we notice that we can not isolate every factor, but due to $\dim \text{Fix}_H \psi_d = 1$ for all d odd and $\dim \text{Fix}_H \chi_2 = 0$ when $H = \langle as \rangle$, using Equation (2.5), and due to $\text{Ind}_H^G 1 = \bigoplus_{\{d: d \in \Omega_{\text{odd}}\}} \mathcal{W}_d$, we know that

$$J_H \sim \prod_{\{d \in \Omega_{\text{odd}}\}} B_d$$

To write $\prod_{\{d \in \Omega_{\text{even}}\}} B_d$ in the decomposition for σ_2 , we notice that $\dim \text{Fix}_H \psi_d = 1$, $d \in \Omega_{\text{even}}$ and $\dim \text{Fix}_H \chi_1 = 0$ and $\dim \text{Fix}_H \psi_d = 0$ for all d odd when $H = \langle a^n, s \rangle$. Using Equation (2.5) again, we know that $J_H \sim \prod_{\{d \in \Omega_{\text{even}}\}} B_d$.

For the second item, we use [7, Corollary 5.6] to prove that this product is a Prym variety of intermediate coverings. Then, since $\text{Ind}_H^G 1 - \text{Ind}_K^G 1 = \chi_2 \bigoplus_{\{d: d \in \Omega_{\text{odd}}\}} \mathcal{W}_d$ and $\dim E_2 = 0$ for σ_2 , we get

that

$$P(X_H/X_K) \sim \prod_{\{d \in \Omega_{\text{odd}}\}} B_d,$$

where $H \leq K$ and $H = \langle s \rangle$, $K = \langle a^n, s \rangle$.

□

5.1.1. *Particular case, $n = p$ a prime number.* When $n = p$ is a prime number, the rational irreducible representations of D_{2p} are χ_0, χ_1, χ_2 and χ_3 of degree one, and two $\mathcal{W}_1, \mathcal{W}_2$ of degree $p - 1$, with associated complex representation ψ_1 and ψ_2 respectively. The character fields of ψ_1, ψ_2 satisfy $[K_d : \mathbb{Q}] = (p - 1)/2$, hence $\deg(\mathcal{W}_d) = \dim_{\mathbb{C}} \psi_d[K_d : \mathbb{Q}] = p - 1$ for $d \in \{1, 2\}$.

Then the group algebra decomposition of JX is given by

$$JX \sim E_0 \times E_1 \times E_2 \times E_3 \times B_1^2 \times B_2^2,$$

where we know that $E_0 = J(X/D_{2p})$ has dimension 0.

Corollary 5.5. *If X is a Riemann surface with action of the group D_{2p} and signature $(0; 2, 2, 2, 2, p)$ then JX has always one elliptic factor in its group algebra decomposition. The dimensions of the factors are given in general by*

Actions	E_0	E_1	E_2	E_3	B_1	B_2
σ_1	0	0	1	0	$p - 1$	0
σ_2	0	1	0	0	$\frac{p-1}{2}$	$\frac{p-1}{2}$

TABLA 5. Dimensions.

In particular, if $p = 3$ we have that, for the action

- (i) σ_1 ; the Jacobian $JX \sim E_2 \times B_1^2$, where E_2 is an elliptic curve and B_1 an abelian surface.
- (ii) σ_2 ; the Jacobian $JX \sim E_1 \times B_1^2 \times B_2^2$, where E_1, B_1 and B_2 are elliptic curves. Hence in this case JX is completely decomposable.

Proof. We obtain the table above directly from Equation 5.2 and Proposition 5.1. The case $p = 3$ is a particular case of this. □

As a direct consequence of Theorem 5.4, we obtain the following corollary.

Corollary 5.6. *If X is a Riemann surface with action of the group D_{2p} and signature $(0; 2, 2, 2, 2, p)$ then the factors of the decomposition of JX in Theorem 5.2 are Jacobian of intermediate coverings or a Prym variety in one case. Indeed we obtain that the decomposition is*

- (i) for the action σ_1 , $JX \sim J(X/\langle a^2 \rangle) \times J(X/\langle s \rangle)^2$.

(ii) for σ_2 , $JX \sim J(X/\langle a^2 \rangle) \times J(X/\langle a^p, s \rangle)^2 \times P(X_{\langle s \rangle}/X_{\langle a^p, s \rangle})^2$.

5.2. Case D_{2n} , n even number. Consider the group D_{2n} and its action as in Sections 3 and 4. Recall that D_{2n} has four complex irreducible representations of degree one, and $n - 1$ of degree two. All of them with Schur index equal to 1.

The rational irreducible representations of D_{2n} are $\chi_0, \chi_1, \chi_2, \chi_3$ of degree one, and

$$\mathcal{W}_d := \mathcal{W}_{\psi_d} = \bigoplus_{\sigma \in \text{Gal}(K_{\psi_d}/\mathbb{Q})} \psi_d^\sigma,$$

for all $d \in \Omega(2n)$.

Then the group algebra decomposition of JX is given by

$$JX \sim E_0 \times E_1 \times E_2 \times E_3 \times \prod_{d \in \Omega(2n)} B_d^2,$$

where E_i are subvarieties of JX associated to the characters χ_i , and we know that $B_0 = J(X/D_{2n})$ is of dimension 0.

In this section we do not include the proofs of the theorems that follow because they are analogous to the proofs corresponding to the action σ_2 in Section 5.1.

Theorem 5.7. *Let n be an even number. If X is a Riemann surface having the unique action of the group D_{2n} with signature $(0; 2, 2, 2, 2, n)$, then JX has always one elliptic factor in its group algebra decomposition. In fact the decomposition is*

$$JX \sim E_1 \times \prod_{\{d \in \Omega(2n)\}} B_d^2$$

where the subindex show the rational representation acting on the factor. E_1 is an elliptic curve and $\dim B_d = \frac{1}{2}\varphi(\frac{2n}{d})$.

As we show in Theorem 5.4, using Lemma 2.2, we have a criterium to write some products of the factors as Jacobian of intermediate coverings.

Theorem 5.8. *Let n be an even number. If X is a Riemann surface having the unique action of the group D_{2n} with signature $(0; 2, 2, 2, 2, n)$. Then JX decomposes as*

$$JX \sim J(X/\langle a^2 \rangle) \times J(X/\langle a^n, s \rangle)^2 \times P(X_{\langle s \rangle}/X_{\langle a^n, s \rangle})^2.$$

6. SHIMURA DOMAINS AND JACOBIANS.

In this section we compute the dimension of the Shimura domains associated to the Jacobian varieties corresponding to the curves with action of $G = D_{2n}$ with signature $(0; 2, 2, 2, 2, n)$. We follow the ideas

in [34, Section 3]. Let X be a Riemann surface of genus g with the action of a finite group G defined by a generating vector. In our case $g = 2n - 1$, $G = D_{2n}$, and we have two non-topologically equivalent actions defined by σ_1 and σ_2 in the case n is odd, and one action σ_2 (up to topological equivalence) if n is even. We show that the Shimura domains have different dimension depending on the action. This is, for n odd the strata $\mathcal{M}(g, G, \theta_1)$ goes via the Jacobi map to a submanifold \mathcal{S}_1 of the moduli space \mathcal{A}_g of principally polarized abelian varieties of dimension N_{n, σ_1} , and for all n the strata $\mathcal{M}(g, G, \theta_2)$ goes to a submanifold \mathcal{S}_2 of dimension N_{n, σ_2} . For n even, the situation is as in $\mathcal{M}(g, G, \theta_2)$ for n odd.

Let us develop some background first. Fix a symplectic basis of the homology of X , this determines a fixed Riemann matrix $Z \in \mathbb{H}_g$ in the Siegel upper half space of complex $g \times g$ symmetric matrices with positive definite imaginary part, of the Jacobian JX of X . This choice also determines a symplectic representation of $L := \text{End}_0(JX) = \text{End}(JX) \otimes_{\mathbb{Z}} \mathbb{Q}$. Every automorphism of X induces a unique automorphism of the corresponding Jacobian, hence G can be considered as a subgroup of the polarization preserving automorphism of JX . Therefore G is isomorphic to the subgroup

$$\Sigma := \{\gamma \in \text{Sp}_{2g}(\mathbb{Z}) : \gamma * Z = Z\},$$

where the action of $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} * Z = (A + ZC)^{-1}(B + ZD).$$

A change of basis induces a different (but equivalent) choice of Z , a conjugated subgroup Σ and a different rational representation of L . Fixing Z it is obtain a submanifold

$$S_G := \{W \in \mathbb{H}_g : \gamma * W = W \text{ for all } \gamma \in \Sigma\}.$$

S_G contains a complex submanifold $\mathbb{H}(L)$ of \mathbb{H}_g parametrizing a *Shimura family* \mathcal{S} of principally polarized abelian varieties containing L in their endomorphism algebras. $\mathbb{H}(L)$ is called [34, Section 3] the *Shimura domain* for Z .

According to [12, Lemma 3.8], the dimension of $\mathbb{H}(L)$ corresponds to

$$N = \dim (S^2(H^{1,0}(S, \mathbb{C})))^G,$$

where $H^{1,0}(S, \mathbb{C})$ is the complex vector space of holomorphic forms. The action of G on X induces a (analytic) representation ρ_a in this vector space. Let $S^2(H^{1,0}(S, \mathbb{C}))$ be the representation of G on the symmetric power of $H^{1,0}(S, \mathbb{C})$, and $(S^2(H^{1,0}(S, \mathbb{C})))^G$ the subspace fixed by G . The dimension of this last subspace can be computed in terms of the character of ρ_a because, by Frobenius reciprocity, it corresponds to the character product of the trivial representation of G with $S^2(H^{1,0}(S, \mathbb{C}))$.

Using Serre's formula [10, Section 1.2]

$$(6.1) \quad N = \frac{1}{2|G|} \sum_{g \in G} (\chi_\rho(g)^2 + \chi_\rho(g^2)),$$

where χ_ρ is the character of ρ_a . Let us write

$$\chi_\rho = \sum_{\chi \in \text{Irr}(G)} \mu_\chi \chi.$$

In general, the coefficients μ_χ can be computed using a classical result due to Chevalley and Weil [8]. In our case, since we know the isotypical decomposition of JX , we know the irreducible representations in ρ_a .

Let us compute detailed these dimensions for the case n a prime number. The general case is similar, but the technical details are harder to be written.

Proposition 6.1. *Let p be a prime number and $D_{2p} = \langle a, b | a^{2p} = s^2 = (as)^2 = 1 \rangle$. Let σ_1 and σ_2 be two generating vectors representing the two non-topologically equivalent actions of D_{2p} with signature $(0; 2, 2, 2, 2, p)$. For $i = 1, 2$, denote by N_{p, σ_i} the dimension of the Shimura domain corresponding to the action determined by σ_i . Then*

$$N_{p, \sigma_1} = \frac{3p-1}{2} \quad \text{and} \quad N_{p, \sigma_2} = p.$$

Proof. Let us denote by ρ_{σ_i} the analytic representation corresponding to the action determined by σ_i . From Theorem 5.2 we have that

$$\rho_{\sigma_1} \equiv \chi_2 \oplus 2W_1 \quad \text{and} \quad \rho_{\sigma_2} \equiv \chi_1 \oplus W_1 \oplus W_2,$$

The character table for these representations is given in Table 7.

Rep.	1	a	a^2	\dots	a^{p-1}	a^p	s	as
\sharp	1	2	2	\dots	2	1	p	p
χ_1	1	1	1	\dots	1	1	-1	-1
χ_2	1	-1	1	\dots	1	-1	1	-1
W_1	$p-1$	1	-1	\dots	1	$-(p-1)$	0	0
W_2	$p-1$	-1	-1	\dots	-1	$(p-1)$	0	0

TABLE 7. Characters to compute the analytic character, prime case.

Using Equation (6.1) to compute N in our context, we have

$$N_{p,\sigma_1} = \frac{1}{8p} \sum_{g \in D_{2p}} ((\chi_2(g) + 2\chi_{W_1}(g))^2 + (\chi_2(g^2) + 2\chi_{W_1}(g^2))).$$

The result follows from replacing the values from Table 7 in the term $((\chi_2(g) + 2\chi_{W_1}(g))^2 + (\chi_2(g^2) + 2\chi_{W_1}(g^2)))$ for each element $g \in G$:

- (1) $g = 1$, $((1 + 2(p-1))^2 + (1 + 2(p-1)))$,
- (2) $g = a^{2j-1}$, $(-1 + 2)^2 + (1 - 2) = 0$,
- (3) $g = a^{2j}$, $(1 - 2)^2 + (1 - 2) = 0$,
- (4) $g = a^p$, $(-1 - 2(p-1))^2 + (1 + 2(p-1))$,
- (5) $g = a^{2j-1}s$, $(1)^2 + (1 + 2(p-1))$,
- (6) $g = a^{2j}s$, $(-1)^2 + (1 + 2(p-1))$.

For the other action we have

$$N_{p,\sigma_2} = \frac{1}{8p} \sum_{g \in D_{2p}} ((\chi_2(g) + \chi_{W_1}(g) + \chi_{W_2}(g))^2 + (\chi_2(g^2) + \chi_{W_1}(g^2) + \chi_{W_2}(g^2))).$$

The result follows in an analogous way as before. Here the terms are

- (1) $g = 1$, $((1 + (p-1) + (p-1))^2 + (1 + (p-1) + (p-1)))$,
- (2) $g = a^{2j-1}$, $(1 + 1 - 1)^2 + (1 - 1 - 1) = 0$,
- (3) $g = a^{2j}$, $(1 - 1 - 1)^2 + (1 - 1 - 1) = 0$,
- (4) $g = a^p$, $(1 - (p-1) + (p-1))^2 + (1 + 2(p-1))$,
- (5) $g = a^{2j-1}s$, $p((-1)^2 + (1 + 2(p-1)))$,
- (6) $g = a^{2j}s$, $p((-1)^2 + (1 + 2(p-1)))$.

□

Remark 6.2. In [12, Thm. 3.9] there is a criterion allowing to prove whether a family of Jacobians contains infinitely many elements with complex multiplication. It is as follows, let G be a finite group acting on a curve of genus g with signature $m = [0; m_1, \dots, m_r]$, and generating vector $\sigma = (c_1, \dots, c_r)$. For a fixed pair (m, σ) , by moving the branch points of the covering in \mathbb{P}^1 one obtains an $(r-3)$ -dimensional family of

such coverings, and a corresponding family of Jacobians $\mathcal{J}(G, m, \sigma)$ of the same dimension. Denote by $Z(G, m, \sigma)$ the closure in the moduli space of principally polarized abelian varieties \mathcal{A}_g of $\mathcal{J}(G, m, \sigma)$, this is the image in \mathcal{A}_g of the connected component of $\mathbb{H}(L)$ containing $\mathcal{J}(G, m, \sigma)$. Let $N = \dim(S^2(H^{1,0}(S, \mathbb{C})))^G$ as before. If N is $(r-3)$, then $Z(G, m, \sigma)$ contains a dense set of Jacobians of CM-type. Notice that our family has the action with 5 branch points, hence it is of dimension 2, and for $p = 2$ we obtain $N_2 = 2$, therefore this family gives infinitely many Jacobian varieties of CM type in dimension $g = 3$. This is the same family (32) in [12, Table 2].

For the general case we have the following result.

Theorem 6.3. *With the notation in Proposition 6.1. Let n be an integer. If n is odd, let σ_1 and σ_2 be the two non-topologically equivalent actions of D_{2n} with signature $(0; 2, 2, 2, 2, n)$. For $i = 1, 2$, denote by N_{n, σ_i} the dimension of the Shimura domain corresponding to the action determined by σ_i . Then*

$$N_{n, \sigma_1} = \frac{3n-1}{2} \quad \text{and} \quad N_{n, \sigma_2} = n.$$

If n is even, let σ_2 be the unique action of D_{2n} with signature $(0; 2, 2, 2, 2, n)$, and denote by N_{n, σ_2} the dimension of the corresponding Shimura domain. Then $N_{n, \sigma_2} = n$.

Proof. As said, the proof follows the same strategy as in Proposition 6.1, but one needs to describe the character of the analytic representation using the decompositions for each case. Let us first review the case n an odd number. In Theorem 5.2 is described the decomposition of the corresponding Jacobian variety, hence the analytic character corresponds to the following for each action. Let us denote them as ρ_{σ_1} and ρ_{σ_2} respectively.

Since n is odd, the set $\Omega(2n)$ decomposes as $\Omega_{\text{even}} \cup \Omega_{\text{odd}}$, as in (5.1). Therefore for each action σ_i we have the following decomposition of the analytic representation

Action	ρ_{σ_i}
σ_1	$\chi_2 \bigoplus_{j \in \Omega_{\text{odd}}} 2\psi_j$
σ_2	$\chi_1 \bigoplus_{j \in \Omega_{\text{even}}} \psi_j \bigoplus_{j \in \Omega_{\text{odd}}} \psi_j$

TABLE 8. Decomposition of the analytic representation.

Denote by $\widetilde{\mathcal{W}}_1 = \bigoplus_{j \in \Omega_{\text{odd}}} \psi_j$ and $\widetilde{\mathcal{W}}_2 = \bigoplus_{j \in \Omega_{\text{even}}} \psi_j$. The character table is given in Table 9.

For n even, we have just one action σ_2 up to topological equivalence. The result follows from computing the dimensions N_{n,σ_1} and N_{n,σ_2} using Equation (6.1).

Rep.	1	a	a^2	\dots	a^{n-1}	a^n	s	as
\sharp	1	2	2	\dots	2	1	n	n
χ_1	1	1	1	\dots	1	1	-1	-1
χ_2	1	-1	$(-1)^2$	\dots	$(-1)^{n-1}$	$(-1)^n$	1	-1
$\widetilde{\mathcal{W}}_1$	$n-1$	1	-1	\dots	1	$-(n-1)$	0	0
$\widetilde{\mathcal{W}}_2$	$n-1$	-1	-1	\dots	-1	$(n-1)$	0	0

TABLE 9. Characters to compute the analytic character.

□

Remark 6.4. In[18] a family \mathcal{F} of principally polarized abelian varieties with action of D_{2p} is studied, for p an odd prime. This family is different from ours because of several reasons; first of all, \mathcal{F} contains no Jacobians [18, Proposition 6.9], the elements in \mathcal{F} have dimension $2p$, and the generic element decomposes as $E_0 \times E_3 \times B_1^2 \times B_2^2$, where the subvarieties are associated to the rational representations of D_{2p} as in Corollary 5.5 and E_i 's are elliptic curves and B_i 's are of dimension $\frac{p-1}{2}$. Finally the Shimura domain for \mathcal{F} has dimension $p+1-$ and ours are of dimension p or $\frac{3p-1}{2}$.

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